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**LETTER TO THE EDITOR**

# Kadomstev–Petviashvile and two-dimensional sine–Gordon equations: reduction to Painlevé transcendents

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**Abstract.** The Kadomstev–Petiviashvile and sine–Gordon equations in two spatial dimensions are shown to be reducible to Painlevé transcendental equations of the first and third kind respectively in terms of suitable Lie-invariant variables.

It is of foremost interest in nonlinear dynamics to analyse the nature of solutions of equations in higher spatial dimensions, whose one-space-dimensional counterparts are solvable by inverse-scattering (IST) methods. It appears to be a basic property (Ablowitz and Segur 1977, Ablowitz *et al* 1978) of IST-solvable one-dimensional systems that on reduction to ordinary differential equations they can be identified with one of the Painlevé transcendental equations (Ince 1956, Davis 1960) with no movable critical points. On the other hand, the Painlevé transcendental equations have themselves been the subject of intensive study recently, and the basic properties of these equations are fairly well understood (Ablowitz and Segur 1977, McCoy *et al* 1977a, b, Rosales 1978, Miles 1978). Thus a certain amount of information about both the partial and the ordinary differential systems could be obtained. In this connection, we investigate whether such nontrivial reductions of higher-dimensional evolution equations to Painlevé-type equations are possible. In particular, we wish to show that two of the important higher-dimensional equations, namely, the Kadomstev–Petviashvile equation in two spatial dimensions (see e.g. Manakov *et al* 1977, Johnson and Thompson 1978),

$$(u_t+6uu_x+u_{xxx})_x+3\alpha^2u_{yy}=0, \tag{1}$$

and the two-dimensional sine–Gordon equation (Leibbrandt 1978),

$$u_{tt}-u_{xx}-u_{yy}+m^2\sin u=0, \tag{2}$$

are reducible to the Painlevé type, the former to the first kind and the latter to the third.

The technique we employ here is that of the Lie method of continuous transformation groups (Bluman and Cole 1974, Lakshmanan and Kaliappan 1979). The general theory says that invariance of a given partial differential equation under a one-parameter Lie group of transformations reduces by one the number of variables appearing in the equation. Then if the reduced system is again Lie-invariant under a new transformation group, further reductions are possible. We employ such a repeated procedure to reduce the systems (1) and (2) to the Painlevé transcendental equations.

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(*i*) *Kadomstev–Petviashvile equation.* Consider the invariance of equation (1) under the infinitesimal transformations

$$x' = x + \epsilon_1 \xi_1, \qquad y' = y + \epsilon_1 \xi_2,$$
$$t' = t + \epsilon_1 \xi_3, \qquad u' = u + \epsilon_1 \xi_4, \tag{3}$$

where $\xi_i \equiv \xi_i(x, y, t, u)$, $i = 1, 2, 3, 4$. One finds the following forms for $\xi_i$,

$$\xi_1 = \tfrac{1}{3}ax + \mu t + \beta + by + dyt,$$
$$\xi_2 = \tfrac{2}{3}ay + c - 6\alpha^2 bt - 3\alpha^2 dt^2, \tag{4}$$
$$\xi_3 = at + \delta, \qquad \xi_4 = -\tfrac{2}{3}au + \tfrac{1}{6}\mu + \tfrac{1}{6}dy,$$

where $a, b, c, d, \mu, \beta$ are arbitrary constants. The invariants of the transformation group of (1) are obtained by solving the Lagrange characteristic equation

$$\mathrm{d}x/\xi_1 = \mathrm{d}y/\xi_2 = \mathrm{d}t/\xi_3 = \mathrm{d}u/\xi_4. \tag{5}$$

We find (5) is exactly solvable with the $\xi_i$'s given by (4), and therefore two similarity variables $\zeta_1$ and $\zeta_2$ can be obtained. For the general case these are quite lengthy, and so in the following we give their form only for the case $d = 0$ in (4). Even when $d \neq 0$ the form of the reduced equation (7) below remains unchanged and hence so do the conclusions. Thus we have (for $d = 0$) the invariant variables

$$\zeta_1 = \frac{a^{-5/3}}{2(at+\delta)^{1/3}}\left[2a^2x - 6aby - 3\left(\mu + \frac{18\alpha^2 b}{a}\right)(at+\delta) - 18\frac{b}{a}(6\alpha^2 b\delta + ac) + 6(a\beta - \mu\delta)\right], \tag{6a}$$

$$\zeta_2 = \frac{a^{-4/3}}{2(at+\delta)^{2/3}}[2a^2y + 36\alpha^2 b(at+\delta) + 3(6\alpha^2 b\delta + ac)], \tag{6b}$$

and

$$F(\zeta_1, \zeta_2) = \frac{3(at+\delta)^{2/3}}{2a^{5/3}}\left(\frac{\mu}{6} - \frac{2au}{3}\right). \tag{6c}$$

Making use of equations (6) in (1), we obtain the reduced partial differential equation

$$3\frac{\partial F}{\partial \zeta_1} + (\zeta_1 + 18F)\frac{\partial^2 F}{\partial \zeta_1^2} + 18\left(\frac{\partial F}{\partial \zeta_1}\right)^2 + 2\zeta_2\frac{\partial^2 F}{\partial \zeta_1\,\partial \zeta_2} - 3\frac{\partial^4 F}{\partial \zeta_1^4} - 9\alpha^2\frac{\partial^2 F}{\partial \zeta_2^2} = 0. \tag{7}$$

Now again treating this as a partial differential equation in the two independent variables $\zeta_1$ and $\zeta_2$, we make another infinitesimal transformation in $\zeta_1$, $\zeta_2$ and $F$,

$$\zeta_1' = \zeta_1 + \epsilon_2\xi_5, \qquad \zeta_2' = \zeta_2 + \epsilon_2\xi_6, \qquad F' = F + \epsilon_2\xi_7, \tag{8}$$

and proceed as before to find the similarity variable

$$\zeta = -9\alpha^2 D\zeta_1 - \tfrac{1}{2}D\zeta_2^2 - E\zeta_2, \tag{9}$$

and

$$\phi(\zeta) = (-162\alpha^2 D)F - \tfrac{1}{2}D\zeta_2^2 + E\zeta_2, \tag{10}$$

where $D, E$ are arbitrary constants. Then in terms of (9) and (10) equation (7) reduces to

$$2187(\alpha^2 D)^3 \frac{d^4\phi}{d\zeta^4} + (\zeta+\phi)\frac{d^2\phi}{d\zeta^2} + \left(\frac{d\phi}{d\zeta}\right)^2 + 2\frac{d\phi}{d\zeta} + 1 = 0. \tag{11}$$

This, after two integrations, becomes

$$\kappa \frac{d^2\phi}{d\zeta^2} + (\phi+\zeta)^2 + p\zeta + q = 0, \tag{12}$$

where $p, q$ are integration constants, and $\kappa = 4374(\alpha^2 D)^3$. With the substitutions

$$\chi = -(\phi+\zeta), \qquad (6\kappa)^{1/2} Z = (\zeta + q/p) \tag{13}$$

equation (12) can be transformed into the form

$$d^2\chi/dZ^2 = 6\chi^2 + \lambda Z, \tag{14}$$

where $\lambda = 6\sqrt{6}\,p$, which is exactly the Painlevé transcendental equation of the first kind (Davis 1960, Ince 1956).

The general solution of equation (14) is characterised (Davis 1960) by the existence of a movable pole with the series solution

$$\chi = a_{-2}/v^2 + a_{-1}/v + a_0 + a_1 v + a_2 v^2 + a_3 v^3 + \dots, \qquad v = Z - Z_0, \tag{15}$$

with

$$a_{-2} = 1, \qquad a_{-1} = 0 = a_0 = a_1, \qquad a_2 = -\tfrac{1}{10}\lambda Z_0, \qquad a_3 = \tfrac{1}{6}\lambda,$$
$$a_4 = h, \qquad a_5 = 0. \tag{16}$$

Here $Z_0$ and $h$ are arbitrary constants. The other coefficients are computed using the recursion formula

$$a_n = \frac{6}{n^2 - n - 12} \sum_{k=-1}^{n-3} a_k a_{n-k-2}, \qquad n > 4. \tag{17}$$

(*ii*) *Two-dimensional sine–Gordon equations.* As before, we make the infinitesimal transformation of the form (4) to equation (2) and find

$$\xi_1 = At + Cy, \qquad \xi_2 = Bt - Cx, \qquad \xi_3 = Ax + By, \qquad \xi_4 = 0, \tag{18}$$

where $A, B, C$ are constants. The corresponding invariants are obtained from the Lagrange equations of the form (5) with $\xi_i$'s given by (18):

$$\zeta_1 = t^2 - x^2 - y^2, \tag{19a}$$

$$\zeta_2 = Ct - Bx + Ay, \tag{19b}$$

and

$$u = F(\zeta_1, \zeta_2). \tag{19c}$$

Then equation (2) reduces to the form

$$4\zeta_1 \frac{\partial^2 F}{\partial \zeta_1^2} + 4\zeta_2 \frac{\partial^2 F}{\partial \zeta_1\, \partial \zeta_2} + 6\frac{\partial F}{\partial \zeta_1} + (C^2 - A^2 - B^2)\frac{\partial^2 F}{\partial \zeta_2^2} + m^2 \sin F = 0. \tag{20}$$

Applying another infinitesimal transformation of the form (8) to the partial differential equation (2), the requirement of invariance results in the similarity variable

$$\zeta = \zeta_1 - \zeta_2^2/(C^2 - A^2 - B^2), \tag{21a}$$

and

$$\phi(\zeta) = F(\zeta_1, \zeta_2). \tag{21b}$$

Then equation (20) reduces to the form

$$4\zeta\frac{d^2\phi}{d\zeta^2} + 4\frac{d\phi}{d\zeta} + m^2 \sin\phi = 0. \tag{22}$$

We make a further transformation

$$w = e^{i\phi}, \tag{23}$$

and so equation (22) reduces to the form

$$\frac{d^2w}{d\zeta^2} = \frac{1}{w}\left(\frac{dw}{d\zeta}\right)^2 - \frac{1}{\zeta}\frac{dw}{d\zeta} + \frac{1}{\zeta}(\alpha w^2 + \beta), \tag{24}$$

where $\alpha = -\beta = \frac{1}{8}m^2$. Equation (24) is the Painlevé transcendental equation of the third kind and has no movable critical points. Detailed analysis of such equations has been done by Painlevé and Gambier and recently by McCoy *et al* (1977a).

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